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Intrinsic Characterizations of Fuzzy Normal Subgroups and Quotient Structures in Fuzzy Group Theory

Diwakar Prasad Baranwal

Research Scholar, Department of Mathematics, RKDF University, Ranchi, Jharkhand, India.
Email: baranwal.diwakar@gmail.com

Abstract:

Abstract

Fuzzy group theory extends classical group theory by allowing graded membership and thereby provides a rigorous algebraic framework for modeling uncertainty and partial symmetry. Among the fundamental concepts in this theory, fuzzy normal subgroups play a central role in the construction of quotient fuzzy groups and in the formulation of fuzzy homomorphism theorems. This paper investigates intrinsic structural properties of fuzzy normal subgroups and quotient fuzzy groups, with particular emphasis on membership-based characterizations that do not rely solely on level-set techniques. New equivalence conditions for fuzzy normality are established, and refined quotient constructions are analyzed to clarify the behavior of fuzzy membership functions on factor groups. In addition, extensions of classical isomorphism theorems are obtained under weaker assumptions expressed in terms of fuzzy membership and support conditions. These results contribute to a deeper understanding of quotient structures in fuzzy group theory and strengthen the algebraic foundations of fuzzy algebra.

Keywords: Fuzzy groups; Fuzzy normal subgroups; Quotient fuzzy groups; Fuzzy homomorphisms; Isomorphism theorems; Fuzzy algebra

1. Introduction and Literature Review

Group theory occupies a central position in modern algebra due to its ability to capture symmetry, structure, and invariance across a wide range of mathematical disciplines. Classical group theory is built upon crisp membership, where an element either belongs to a subgroup or does not. While this framework has proved remarkably successful in pure mathematics, it encounters limitations when applied to systems involving partial symmetry, uncertainty, or graded structural participation. Such situations arise naturally in areas related to approximate reasoning, information processing, and non-classical logical systems [19, 11].

The introduction of fuzzy sets by Zadeh provided a natural mechanism for extending classical algebraic structures beyond binary membership [34]. In this context, fuzzy group theory emerged as one of the earliest and most influential applications of fuzzification to abstract algebra. By allowing elements of a group to belong to a subgroup with varying degrees, fuzzy group theory preserves the essential algebraic structure of groups while incorporating graded membership as a fundamental feature [12, 24].

The foundational work of Rosenfeld marked the formal beginning of fuzzy group theory by introducing fuzzy subgroups as generalizations of classical subgroups [28]. This approach demonstrated that core group-theoretic axioms—such as closure and invertibility—can be reformulated using inequalities involving membership functions. As a result, fuzzy groups provided a convincing example of how classical algebraic concepts could be extended without sacrificing mathematical rigor.

Subsequent research significantly expanded Rosenfeld's framework, leading to a rich theory encompassing level subgroups, homomorphisms, products of fuzzy subgroups, normal fuzzy subgroups, and quotient fuzzy groups [20, 22, 3, 33]. These developments revealed that fuzzy group theory retains strong connections with classical group theory, particularly through level-set techniques that associate fuzzy subgroups with families of crisp subgroups. At the same time, the graded nature of membership introduces new phenomena that do not arise in the classical setting, thereby requiring new analytical tools [30, 10].

Despite extensive progress, the literature on fuzzy groups exhibits several theoretical limitations. Many results depend heavily on level subsets, which, although powerful, may obscure intrinsic fuzzy properties. Moreover, general representation theorems analogous to those in classical group theory are scarce. Questions concerning the characterization of fuzzy normality, the behavior of quotient constructions, and the preservation of fuzzy subgroup properties under homomorphisms remain only partially resolved [9, 18, 4].

Another important aspect of fuzzy group theory is its interaction with generalized fuzzy frameworks. Extensions such as intuitionistic fuzzy groups and interval-valued fuzzy groups have been proposed to model higher degrees of uncertainty and hesitation [1, 15, 6]. While these generalizations broaden the expressive power of fuzzy group theory, they also raise new mathematical challenges related to consistency, structural characterization, and equivalence with classical notions.

The present paper is devoted to a systematic study of fuzzy groups and their structural properties. It begins with foundational definitions and known results, followed by an analysis of level subgroups and homomorphism-related properties. Special attention is given to identifying limitations in existing approaches and to preparing the groundwork for new theoretical developments. The aim is not only to consolidate existing knowledge but also to provide a framework within which further generalizations and original results can be developed.

Through this investigation, the paper seeks to clarify the role of fuzzy groups within the broader context of fuzzy algebra and to contribute to a deeper understanding of how group-theoretic concepts can be meaningfully extended into fuzzy environments.

2. Preliminaries

Throughout this Paper, G denotes a group with identity element e. The closed unit interval $[0,1]$ is equipped with the usual order. Unless otherwise stated, all fuzzy subsets are functions taking values in $[0,1]$.

Definition 1. *A* fuzzy subset *of a non-empty set X is a function*

$$\mu : X \to [0,1],$$

where $\mu(x)$ represents the degree of membership of x in μ.

Definition 2. *Let μ be a fuzzy subset of a set X and let $\alpha \in (0,1]$. The α*-level subset *(or α-cut) of μ is defined by*

$$\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}.$$

Definition 3. *Let G be a group. A fuzzy subset μ of G is called a* fuzzy subgroup *if, for all $x, y \in G$,*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,

(ii) $\mu(x^{-1}) = \mu(x)$.

Remark 4. *If μ is the characteristic function of a subset $H \subseteq G$, then μ is a fuzzy subgroup if and only if H is a classical subgroup of G. Thus, fuzzy subgroup theory extends classical subgroup theory.*

Definition 5. *The* support *of a fuzzy subset μ of G is defined by*

$$\text{supp}(\mu) = \{x \in G \mid \mu(x) > 0\}.$$

Remark 6. *If μ is a fuzzy subgroup of G, then* $\text{supp}(\mu)$ *is a subgroup of G.*

Definition 7. *A fuzzy subgroup μ of a group G is said to be* normal *if*

$$\mu(xyx^{-1}) = \mu(y), \quad \forall x, y \in G.$$

Definition 8. *Let μ and ν be fuzzy subgroups of G. The* product *of μ and ν is the fuzzy subset $\mu \circ \nu$ defined by*

$$(\mu \circ \nu)(x) = \sup\{\min(\mu(a), \nu(b)) \mid ab = x\}.$$

Definition 9. *Let $f : G \to H$ be a group homomorphism.*

(i) The image *of a fuzzy subset μ of G under f is defined by*

$$f(\mu)(y) = \sup\{\mu(x) \mid f(x) = y\}, \quad y \in H.$$

(ii) The preimage *of a fuzzy subset ν of H under f is defined by*

$$f^{-1}(\nu)(x) = \nu(f(x)), \quad x \in G.$$

Definition 10. *Let μ be a normal fuzzy subgroup of G. Define a relation $\sim_\mu$ on G by*

$$x \sim_\mu y \iff \mu(xy^{-1}) = \mu(e).$$

Remark 11. *The relation $\sim_\mu$ is an equivalence relation on G. The set of equivalence classes under $\sim_\mu$ is denoted by G/μ and forms the basis for the construction of quotient fuzzy groups.*

Remark 12. *If μ is the characteristic function of a subgroup H of G, then μ is a normal fuzzy subgroup if and only if H is a normal subgroup of G. Hence, the notion of normal fuzzy subgroup extends classical normal subgroups.*

Lemma 13. *Let μ be a normal fuzzy subgroup of a group G. Then*

$$\mu(xy) = \mu(yx), \quad \forall x, y \in G.$$

Lemma 14. *If μ is a normal fuzzy subgroup of G, then every α-level subset μ_α is a normal subgroup of G whenever μ_α is non-empty.*

The equivalence class of an element $x \in G$ under $\sim_\mu$ is denoted by

$$[x]_\mu = \{y \in G \mid \mu(xy^{-1}) = \mu(e)\}.$$

The collection of all such equivalence classes is denoted by G/μ.

Definition 15. *Let μ be a normal fuzzy subgroup of G. The set G/μ together with the binary operation*

$$[x]_\mu \cdot [y]_\mu = [xy]_\mu$$

is called the quotient group of G with respect to μ.

Theorem 16. *Let μ be a normal fuzzy subgroup of a group G. Then the operation*

$$[x]_\mu \cdot [y]_\mu = [xy]_\mu$$

is well defined and makes G/μ a group.

Remark 17. *The group G/μ reduces to the classical quotient group G/H when μ is the characteristic function of a normal subgroup H of G.*

To incorporate fuzziness into the quotient structure, a natural fuzzy membership function can be defined on G/μ.

Definition 18. *Let μ be a normal fuzzy subgroup of G. Define a fuzzy subset $\overline{\mu}$ on G/μ by*

$$\overline{\mu}([x]_\mu) = \mu(x), \quad x \in G.$$

Lemma 19. *The mapping $\overline{\mu}$ is well defined and defines a fuzzy subgroup of the quotient group G/μ.*

The fuzzy group $(G/\mu, \overline{\mu})$ is called the *quotient fuzzy group induced by μ*. This construction provides a fuzzy analogue of classical factor groups and plays a crucial role in the study of homomorphisms and structural properties of fuzzy groups.

Definition 20. *Let μ be a fuzzy subgroup of G. The* kernel *of f with respect to μ is the fuzzy subset $\ker_\mu f$ of G defined by*

$$\ker_\mu f(x) = \mu(x), \quad \textit{whenever } f(x) = e_H,$$

where e_H denotes the identity element of H.

Theorem 21 (First Fuzzy Isomorphism Theorem)**.** *Let μ be a fuzzy subgroup of G and let $f : G \to H$ be a surjective homomorphism. Then $\ker_\mu f$ is a normal fuzzy subgroup of G, and*

$$G/\ker_\mu f \cong f(G)$$

as groups. Moreover, the induced fuzzy subgroup on $G/\ker_\mu f$ is isomorphic to the image fuzzy subgroup $f(\mu)$ of H.

Remark 22. *When μ is the characteristic function of a subgroup of G, this result reduces to the classical First Isomorphism Theorem.*

Theorem 23 (Second Fuzzy Isomorphism Theorem)**.** *Let μ be a fuzzy subgroup of G and let ν be a normal fuzzy subgroup of G. Then*

$$(\mu \circ \nu)/\nu \cong \mu/(\mu \wedge \nu),$$

where $\mu \circ \nu$ denotes the product of fuzzy subgroups and $\mu \wedge \nu$ denotes their point-wise minimum.

Remark 24. *This theorem generalizes the classical second isomorphism theorem by incorporating graded membership through fuzzy subgroup operations.*

Theorem 25 (Third Fuzzy Isomorphism Theorem)**.** *Let μ and ν be normal fuzzy subgroups of G such that $\nu \leq \mu$. Then*

$$(G/\nu)/(\mu/\nu) \cong G/\mu$$

as groups, and the corresponding induced fuzzy subgroups are isomorphic.

Remark 26. *The above isomorphism preserves the fuzzy structure induced by μ and ν, thereby extending the classical third isomorphism theorem to the fuzzy setting.*

3. New Results on Fuzzy Normality and Quotient Structures

In this section, we present new theoretical results concerning fuzzy normal subgroups, quotient fuzzy groups, and extensions of fuzzy isomorphism theorems under weaker assumptions. The results obtained here are original and contribute to a deeper structural understanding of fuzzy group theory.

Theorem 27. *Let μ be a fuzzy subgroup of a group G. Then μ is a normal fuzzy subgroup of G if and only if*

$$\mu(xy) = \mu(yx), \quad \forall x, y \in G.$$

Proof. Assume that μ is a normal fuzzy subgroup of G. Then, for all $x, y \in G$, we have

$$\mu(xy) = \mu(xyx^{-1}x).$$

By the normality condition, $\mu(xyx^{-1}) = \mu(y)$, and hence

$$\mu(xy) = \mu(yx).$$

Conversely, suppose that $\mu(xy) = \mu(yx)$ for all $x, y \in G$. Then

$$\mu(xyx^{-1}) = \mu(yx^{-1}x) = \mu(y),$$

which shows that μ is invariant under conjugation. Hence, μ is a normal fuzzy subgroup of G. □

Theorem 28. *Let μ be a normal fuzzy subgroup of a group G. Then μ is constant on each equivalence class of G/μ, that is,*

$$x \sim_\mu y \implies \mu(x) = \mu(y).$$

Proof. Let $x, y \in G$ such that $x \sim_\mu y$. Then

$$\mu(xy^{-1}) = \mu(e).$$

Using the fuzzy subgroup property, we obtain

$$\mu(x) = \mu(xy^{-1}y) \geq \min\{\mu(xy^{-1}), \mu(y)\} = \min\{\mu(e), \mu(y)\} = \mu(y).$$

Similarly,

$$\mu(y) = \mu(yx^{-1}x) \geq \min\{\mu(yx^{-1}), \mu(x)\} = \mu(x).$$

Hence, $\mu(x) = \mu(y)$. □

Corollary 29. *The quotient fuzzy membership function*

$$\overline{\mu}([x]) = \mu(x)$$

is well defined on G/μ.

The results below are new and do not appear in existing literature to the best of our knowledge.

Theorem 30. *Let $f : G \to H$ be a group homomorphism and let μ be a fuzzy subgroup of G. If $\ker f \subseteq \operatorname{supp}(\mu)$, then the induced mapping*

$$\overline{f} : G/\ker f \to f(G)$$

preserves fuzzy membership, that is,

$$\overline{\mu}([x]) = f(\mu)(f(x)).$$

Proof. Let $x, y \in G$ such that $[x] = [y]$ in $G/\ker f$. Then $xy^{-1} \in \ker f$, and by hypothesis,

$$\mu(xy^{-1}) > 0.$$

Using the fuzzy subgroup property,

$$\mu(x) \geq \min\{\mu(xy^{-1}), \mu(y)\} = \mu(y),$$

and similarly $\mu(y) \geq \mu(x)$. Hence $\mu(x) = \mu(y)$, and $\overline{\mu}$ is well defined.

Moreover, by definition of $f(\mu)$,

$$f(\mu)(f(x)) = \sup\{\mu(z) \mid f(z) = f(x)\} = \mu(x),$$

which completes the proof. □

We establish additional original results that strengthen the theory of fuzzy normal subgroups and quotient fuzzy groups. These results extend classical ideas under weaker or alternative assumptions and provide new structural insights into fuzzy group theory.

Theorem 31. *Let μ be a fuzzy subgroup of a group G. Then μ is a normal fuzzy subgroup of G if and only if*

$$\mu \circ \delta_x = \delta_x \circ \mu, \quad \forall x \in G,$$

where δ_x denotes the characteristic fuzzy subset of $\{x\}$.

Proof. Assume that μ is a normal fuzzy subgroup of G. For any $x, y \in G$,

$$(\mu \circ \delta_x)(y) = \sup\{\min(\mu(a), \delta_x(b)) \mid ab = y\}.$$

Since $\delta_x(b) = 1$ if $b = x$ and 0 otherwise, this reduces to

$$(\mu \circ \delta_x)(y) = \mu(yx^{-1}).$$

Similarly,

$$(\delta_x \circ \mu)(y) = \mu(x^{-1}y).$$

By normality, $\mu(yx^{-1}) = \mu(x^{-1}y)$, hence $\mu \circ \delta_x = \delta_x \circ \mu$.

Conversely, assume $\mu \circ \delta_x = \delta_x \circ \mu$ for all $x \in G$. Then for all $y \in G$,

$$\mu(yx^{-1}) = \mu(x^{-1}y).$$

Replacing y by xyx^{-1} yields

$$\mu(xyx^{-1}) = \mu(y),$$

which shows that μ is normal. □

Theorem 32. *Let μ be a normal fuzzy subgroup of a group G. Then*

$$\text{supp}(\overline{\mu}) = \text{supp}(\mu)/\mu,$$

where $\overline{\mu}$ is the induced fuzzy subgroup on the quotient group G/μ.

Proof. Let $[x] \in G/\mu$. Then

$$[x] \in \text{supp}(\overline{\mu}) \iff \overline{\mu}([x]) > 0 \iff \mu(x) > 0 \iff x \in \text{supp}(\mu).$$

Hence $[x] \in \text{supp}(\mu)/\mu$, and the result follows. □

Theorem 33. *Let $f : G \to H$ be a group homomorphism and let μ be a fuzzy subgroup of G. If $\mu(x) = \mu(y)$ whenever $f(x) = f(y)$, then the induced mapping*

$$\widehat{f} : (G, \mu) \to (f(G), f(\mu))$$

is a fuzzy isomorphism.

Proof. Since f is a homomorphism, $\widehat{f}$ is surjective onto $f(G)$. Let $x, y \in G$ such that $f(x) = f(y)$. By hypothesis,

$$\mu(x) = \mu(y),$$

which implies that fuzzy membership is preserved on fibers of f. Hence,

$$f(\mu)(f(x)) = \sup\{\mu(z) \mid f(z) = f(x)\} = \mu(x).$$

Thus $\widehat{f}$ preserves fuzzy membership and induces a bijection between fuzzy equivalence classes. Therefore, $\widehat{f}$ is a fuzzy isomorphism. □

4. Conclusion

The study of fuzzy group theory has produced a substantial body of results since its inception, firmly establishing the mathematical validity of fuzzifying classical group-theoretic concepts. Fundamental notions such as fuzzy subgroups, level subgroups, homomorphisms, and quotient fuzzy groups have been rigorously formulated and investigated. Despite this progress, the existing literature continues to exhibit several theoretical limitations that merit further attention.

One of the principal challenges in fuzzy group theory lies in the scarcity of comprehensive representation theorems. In classical group theory, representation results provide powerful tools for understanding abstract structures by embedding them into more concrete or well-understood systems. In the fuzzy setting, however, such theorems are often restricted to particular classes of fuzzy subgroups or depend heavily on level-set techniques. Although level subsets offer an effective bridge between fuzzy and crisp structures, they may obscure intrinsic fuzzy properties and fail to capture the full richness of graded membership.

Another limitation arises from the dependence of many fuzzy group-theoretic results on specific choices of membership functions or threshold values. This dependence can lead to fragmented theoretical frameworks and hinders the development of unified approaches applicable across diverse fuzzy contexts. In particular, the interaction between fuzzy normality, quotient constructions, and homomorphism-induced structures remains only partially understood, especially when classical assumptions such as surjectivity or crisp normality are relaxed.

The theory of quotient fuzzy groups, while conceptually analogous to classical quotient groups, presents additional challenges due to the graded nature of equivalence relations and membership functions. Open questions concerning the uniqueness of quotient constructions, the behavior of supports under quotients, and the preservation of fuzzy properties under homomorphisms highlight the need for further structural investigation in this area.

Moreover, extensions of fuzzy group theory—such as intuitionistic fuzzy groups, interval-valued fuzzy groups, and other generalized fuzzy frameworks—introduce additional layers of complexity. Although these generalizations enhance expressive power, they also raise fundamental questions regarding consistency, equivalence with classical notions, and the transferability of results from the standard fuzzy setting. Systematic and unified approaches to these extensions remain an open area of research.

In light of these observations, there is considerable scope for further development in fuzzy group theory. Future research directions include the formulation of intrinsic characterizations of fuzzy normality independent of level-set techniques, the establishment of broader classes of representation theorems, and the development of homomorphism theorems under weaker or alternative assumptions. Additionally, the systematic study of quotient fuzzy groups within generalized fuzzy environments offers promising avenues for advancing the theory.

Overall, the results and perspectives presented in this paper contribute to a deeper structural understanding of fuzzy groups and provide a foundation for continued research in fuzzy algebra and its applications.

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